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Structure Matrices of Algebras

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1. INTRODUCTION

Let F be any field and S any finite semigroup, say of order m, and let $F[S]$ denote the semigroup algebra of S over F. In an earlier article[1] [3] we observed that the (nil) radical N (and hence also the semisimplicity) of $A = F[S]$ can (with some restrictions on the characteristic of F) be very conveniently described and/or computed by means of two symmetric $m \times m$ "structure matrices" L, R with integer entries, and that these L, R also yield information about A/N.

As indicated in [3], the same methods and results extend to *arbitrary* finite-dimensional associative F-algebras A (with L, R still symmetric, but now with entries in F), and we present the details here, largely in the form of a repertoire of procedures which, inter alia, when the characteristic of F is either zero or sufficiently large, greatly facilitate the task of computing the radicals both of general finite-dimensional F-algebras and of various combinations (e.g., tensor products) of such algebras. Thus several of our results are to the effect that, by use of L, R (which are themselves easily computable), certain structures related to a given algebra or algebras can be concretely determined by easy matrix procedures. Such results typically present themselves as explicit formulae (of which some have consequences which might seem less obvious without the use of structure matrices). We also show (Theorem 4) that $L = R$ for every Frobenius (in particular, every semisimple) algebra, and note some analogies with Lie algebra theory.

The two main limitations of the structure matrix approach presented here are, first, that L, R determine the radical N of A only when the characteristic of F is either zero or a prime $p > \dim_F A$, and, second, that, even in zero characteristic, L, R do not in general contain enough information to

[1] We take this opportunity to note that, on p. 271 of [3], the conclusion of Lemma 1 should read $c^{-1}b = b$; also, in line 10 of p. 276, read "some" instead of "every."

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determine all the structure of A, or even of A/N (see Propositions 9 and 10). However, in view of Theorems 1, 2, and 3 below, it would not be surprising to learn that some other interesting properties of (or constructions on) algebras can be characterized in terms of L, R. To the extent that such characterizations may prove feasible, one could thus reduce certain problems in the theory of associative F-algebras $A, B, \ldots$ to problems about symmetric matrices $(L_A, R_A), (L_B, R_B), \ldots$ over F.

2. The Structure Matrices L, R

Throughout, F will denote a given field, and A an arbitrary finite-dimensional associative algebra over F (we do not require that A have any unity element). Given any basis $\mathscr{A} = \{a_1, \ldots, a_m\}$ of A over F, this determines m^3 "structure constants" $\gamma_{ij}^k \in F$ $(i, j, k = 1, \ldots, m)$ such that

$$a_i a_j = \sum_k \gamma_{ij}^k a_k \qquad (i, j = 1, \ldots, m).$$

We refer to the pair $(A, \mathscr{A})$, together with the corresponding γ_{ij}^k, as a *presentation* of A. For any such presentation, we introduce $2m^2$ corresponding scalars

$$\lambda_{ij} = \lambda_{ij}^{\mathscr{A}} = \sum_{k,t} \gamma_{ij}^k \gamma_{tk}^t, \qquad \rho_{ij} = \rho_{ij}^{\mathscr{A}} = \sum_{k,t} \gamma_{ji}^k \gamma_{kt}^t,$$

where (now and hereafter) the summations extend independently over the range $1, \ldots, m$. We thus have two $m \times m$ matrices

$$L = L^{\mathscr{A}} = L_{A,F}^{\mathscr{A}} = (\lambda_{ij}), \qquad R = R^{\mathscr{A}} = R_{A,F}^{\mathscr{A}} = (\rho_{ij}),$$

called the *left* and *right structure matrices* of A with respect to $\mathscr{A}$.

The motivation for these definitions of λ_{ij}, ρ_{ij}, L, R may not be immediately apparent, but this mystery will be resolved in Section 4, where $L^{\mathscr{A}}, R^{\mathscr{A}}$ turn out to be just the matrices, relative to $\mathscr{A}$, of two invariantly definable ("right and left trace") bilinear forms on A. However, for the sake of demonstrating the roles of L, R in explicit computations, we shall work mainly from a non-invariant point of view.

Proposition 1. *For all $F, A, \mathscr{A}$, the matrices L, R are both always symmetric.*

Proof. For any h, i, j, we have

$$a_h(a_i a_j) = a_h \sum_k \gamma_{ij}^k a_k = \sum_k \gamma_{ij}^k \sum_t \gamma_{hk}^t a_t,$$

while

$$(a_h a_i)\, a_j = \left(\sum_s \gamma_{hi}^s a_s\right) a_j = \sum_s \gamma_{hi}^s \sum_t \gamma_{sj}^t a_t,$$

so, by associativity,

$$\sum_k \gamma_{ij}^k \gamma_{hk}^t = \sum_s \gamma_{hi}^s \gamma_{sj}^t \qquad (h, i, j, t = 1,..., m).$$

In particular, on taking $h = t$, it follows that

$$\lambda_{ij} = \sum_t \left(\sum_k \gamma_{ij}^k \gamma_{tk}^t\right) = \sum_t \sum_s \gamma_{ti}^s \gamma_{sj}^t \qquad (i, j = 1,..., m),$$

which is obviously symmetric under the interchange of i with j.

Similarly, $\rho_{ij} = \rho_{ji}$ for all relevant i, j. ∎

It is often computationally desirable to choose $\mathscr{A}$ so that L and/or R takes some "simple" form, while also theoretical considerations (see Sections 5 and 6) sometimes require a knowledge of how the matrices $L^{\mathscr{A}}$, $R^{\mathscr{A}}$ depend on the choice of basis $\mathscr{A}$. In this connection, we have

PROPOSITION 2. *If $\mathscr{A} = \{a_1,..., a_m\}$ and $\mathscr{C} = \{c_1,..., c_m\}$ are any two bases for A over F, say with*

$$c_p = \sum_i \pi_{pi} a_i \qquad (p = 1,..., m)$$

(where each $\pi_{pi} \in F$), then

$$L^{\mathscr{C}} = PL^{\mathscr{A}}P', \qquad R^{\mathscr{C}} = PR^{\mathscr{A}}P',$$

where the $m \times m$ matrix P is given by $P = (\pi_{pi})$ and P' denotes the transpose of P.

Proof. Write also $a_i = \sum_p \phi_{ip} c_p$, so that $\sum_p \phi_{ip} \pi_{pj} = \delta_{ij}$, the Kronecker symbol. Then we have

$$c_p c_q = \left(\sum_i \pi_{pi} a_i\right)\left(\sum_j \pi_{qj} a_j\right) = \sum_{i,j} \pi_{pi} \pi_{qj} \sum_k \gamma_{ij}^k a_k = \sum_r \eta_{pq}^r c_r,$$

where

$$\eta_{pq}^r = \sum_{i,j,k} \pi_{pi} \pi_{qj} \gamma_{ij}^k \phi_{kr}.$$

Hence

$$\lambda_{pq}^{\mathscr{C}} = \sum_{r,v} \eta_{pq}^{r} \eta_{vr}^{v} = \sum_{r,v} \left(\sum_{i,j,k} \pi_{pi} \pi_{qj} \gamma_{ij}^{k} \phi_{kr} \right) \left(\sum_{x,y,z} \pi_{vx} \pi_{ry} \gamma_{xy}^{z} \phi_{zv} \right)$$

$$= \sum_{i,j,k,x,y,z} \pi_{pi} \pi_{qj} \gamma_{ij}^{k} \gamma_{xy}^{z} \delta_{ky} \delta_{zx} = \sum_{i,j} \pi_{pi} \lambda_{ij}^{\mathscr{A}} \pi_{qj} = (PL^{\mathscr{A}}P')_{pq}.$$

Similarly, $R^{\mathscr{C}} = PR^{\mathscr{A}}P'$. ∎

Thus, by the elementary theory of quadratic forms (i.e., "completing squares"), we have

COROLLARY. *If F does not have characteristic 2, then, for any given algebra A over F, we can find a basis $\mathscr{B}$ such that $L^{\mathscr{B}}$ is diagonal (and also a basis $\mathscr{C}$ such that $R^{\mathscr{C}}$ is diagonal).*

When F has characteristic 2 we can still diagonalize $L^{\mathscr{A}}$ provided that $L^{\mathscr{A}}$ has at least one nonzero diagonal entry (see e.g., [4, pp. 103–104, Theorem 3]); and possibly even this proviso is not essential, i.e., perhaps cases such as $L = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ cannot occur.

I do not know whether L, R can always be simultaneously diagonalized (even when F is the real or complex field).

We compute the matrices L, R explicitly in two important cases:

PROPOSITION 3. *Let $M_n(F)$ denote the F-algebra of all $n \times n$ matrices over F, and let $\mathscr{E} = \{e_{ij}: i,j = 1,\dots, n\}$ denote its standard basis. Then*

$$L^{\mathscr{E}} = R^{\mathscr{E}} = (\lambda_{(i,j),(r,s)}^{\mathscr{E}}) = n(\delta_{is}\delta_{jr}).$$

Proof. Here $e_{ij}e_{rs} = \delta_{jr}e_{is} = \delta_{jr}\sum_{p,q}\delta_{ip}\delta_{qs}e_{pq}$, and so, on labelling $\mathscr{E}$ by the pairs (i,j), we have corresponding structure constants $\gamma_{(i,j),(r,s)}^{(p,q)} = \delta_{jr}\delta_{ip}\delta_{qs}$, and

$$\lambda_{(i,j),(r,s)}^{\mathscr{E}} = \sum_{p,q,u,v} \gamma_{(i,j),(r,s)}^{(p,q)} \gamma_{(u,v),(p,q)}^{(u,v)}$$

$$= \sum_{p,q,u,v} \delta_{jr}\delta_{ip}\delta_{qs} \cdot \delta_{vp}\delta_{uu}\delta_{vq} = n\delta_{jr}\delta_{is}.$$

Similarly,

$$\rho_{(i,j),(r,s)}^{\mathscr{E}} = n\delta_{is}\delta_{jr}. \quad ∎$$

PROPOSITION 4. *Let $\mathbb{H}$ denote the quaternion algebra over the reals, with basis $\mathscr{H} = \{1, i, j, k\}$, subject to the usual multiplication rules*

$$i^2 = j^2 = k^2 = -1, \qquad jk = -kj = i, \qquad ki = -ik = j, \qquad ij = -ji = k.$$

Then $L^{\mathscr{H}} = R^{\mathscr{H}} = 4\,\mathrm{diag}(1, -1, -1, -1)$.

The details of the verification are left to the reader.

The fact that $L = R$ in Propositions 3 and 4 is not accidental (see Theorem 4 below), but it is easy to construct (e.g., 2-dimensional) algebras with $L \neq R$.

3. BEHAVIOR OF L, R UNDER TENSOR PRODUCT, DIRECT SUM, ETC.

As background for the discussions of Sections 4 and 5, we next examine how L, R transform under various standard constructions and procedures.

We consider any pair of F-algebras, say A with basis $\mathscr{A} = \{a_1,..., a_m\}$ and γ_{ij}^k as before, and now also B with basis $\mathscr{B} = \{b_1,..., b_n\}$ and structure constants ε_{pq}^r, i.e.,

$$b_p b_q = \sum_r \varepsilon_{pq}^r b_r \qquad (p, q = 1,..., n).$$

Every index of summation will range over the set $1,..., m$ or the set $1,..., n$; it will always be obvious from the context which is the relevant range for each index, and we shall not indicate the ranges explicitly.

PROPOSITION 5. *Relative to the basis*

$$\mathscr{C} = \mathscr{A} \otimes \mathscr{B} = \{a_i \otimes b_p : i = 1,..., m; p = 1,..., n\},$$

the F-algebra $C = A \otimes_F B$ has

$$L^{\mathscr{C}} = L^{\mathscr{A}} \otimes L^{\mathscr{B}}, \qquad R^{\mathscr{C}} = R^{\mathscr{A}} \otimes R^{\mathscr{B}}$$

(where, on the right-hand sides, $\otimes$ denotes the Kronecker product of matrices).

Proof. To compute $L^{\mathscr{C}}, R^{\mathscr{C}}$, we must first obtain the structure constants $\eta_{(i,p),(j,q)}^{(k,r)}$ of the basis $\mathscr{C}$:

$$\begin{aligned}(a_i \otimes b_p)(a_j \otimes b_q) &= (a_i a_j) \otimes (b_p b_q)\\ &= \Big(\sum_k \gamma_{ij}^k a_k\Big) \otimes \Big(\sum_r \varepsilon_{pq}^r b_r\Big) = \sum_{k,r} \gamma_{ij}^k \varepsilon_{pq}^r (a_k \otimes b_r),\end{aligned}$$

i.e.,

$$\eta^{(k,r)}_{(i,p),(j,q)} = \gamma^k_{ij}\varepsilon^r_{pq}.$$

Hence

$$\lambda^{\mathscr{C}}_{(i,p),(j,q)} = \sum_{k,r,t,u} \eta^{(k,r)}_{(i,p)(j,q)}\eta^{(t,u)}_{(t,u),(k,r)}$$
$$= \sum_{k,r,t,u} \gamma^k_{ij}\varepsilon^r_{pq}\cdot\gamma^t_{tk}\varepsilon^u_{ur} = \lambda^{\mathscr{A}}_{ij}\lambda^{\mathscr{B}}_{pq}.$$

Similarly, $\rho^{\mathscr{C}}_{(i,p),(j,q)} = \rho^{\mathscr{A}}_{ij}\rho^{\mathscr{B}}_{pq}$. ∎

PROPOSITION 6. *Relative to the basis*

$$\mathscr{C} = \mathscr{A} \cup \mathscr{B} = \{a_1,..., a_m, b_1,..., b_n\} = \{c_1,..., c_{m+n}\},$$

the F-algebra $C = A \oplus B$ *has*

$$L^{\mathscr{C}} = L^{\mathscr{A}} \oplus L^{\mathscr{B}} = \begin{pmatrix} L^{\mathscr{A}} & 0 \\ 0 & L^{\mathscr{B}} \end{pmatrix}, \qquad R^{\mathscr{C}} = R^{\mathscr{A}} \oplus R^{\mathscr{B}} = \begin{pmatrix} R^{\mathscr{A}} & 0 \\ 0 & R^{\mathscr{B}} \end{pmatrix}.$$

The proof of Proposition 4 is computational and is omitted.

PROPOSITION 7. *Let A, C be given F-algebras, and let*

$$\theta: A \to C$$

be a given surjective F-algebra homomorphism, with kernel B. Also let $\mathscr{B} = \{b_1,..., b_n\}$ *be a basis of B, and extend this to form a basis* $\mathscr{A} = \{a_1,..., a_m, b_1,..., b_n\}$ *of A, where, for suitable* $\gamma^k_{ij}, \zeta^r_{ij} \in F$,

$$a_i a_j = \sum_k \gamma^k_{ij} a_k + \sum_r \zeta^r_{ij} b_r \qquad (i, j = 1,..., m).$$

Then $\mathscr{C} = \{a_1\theta,..., a_m\theta\}$ *is a basis of C, and*

$$L^{\mathscr{C}} = (\lambda^{\mathscr{C}}_{ij}) = \Big(\sum_{k,t} \gamma^k_{ij}\gamma^t_{tk}\Big), \qquad R^{\mathscr{C}} = (\rho^{\mathscr{C}}_{ij}) = \Big(\sum_{k,t} \gamma^k_{ji}\gamma^t_{kt}\Big).$$

Proof. That $\mathscr{C}$ is a basis of C is trivial. Also

$$(a_i\theta)(a_j\theta) = (a_i a_j)\theta = \Big(\sum_k \gamma^k_{ij} a_k + \sum_r \zeta^r_{ij} b_r\Big)\,\theta = \sum_k \gamma^k_{ij}(a_k\theta),$$

so $\lambda^{\mathscr{C}}_{ij} = \sum_{k,t} \gamma^k_{ij}\gamma^t_{tk}$, and similarly for $\rho^{\mathscr{C}}_{ij}$. ∎

As a matter of notation, we may (after arranging that $\mathscr{B} \subseteq \mathscr{A}$) write $L^{\mathscr{C}} = L^{\mathscr{A}(\mathrm{mod}\,\mathscr{B})}$, $R^{\mathscr{C}} = R^{\mathscr{A}(\mathrm{mod}\,\mathscr{B})}$.

PROPOSITION 8. *Let B be a field, F a subfield of B, and A any B-algebra. Let $\mathscr{A} = \{a_1,..., a_m\}$ be a basis for A over B, and $\mathscr{B} = \{b_1,..., b_n\}$ a basis for B over F.*

Then $\mathscr{C} = \mathscr{A}\mathscr{B} = \{a_i b_p: i = 1,..., m; p = 1,..., n\}$ is a basis for A over F, and

$$L^{\mathscr{C}}_{A,F} = L^{\mathscr{A}}_{A,B} \otimes L^{\mathscr{B}}_{B,F}, \qquad R^{\mathscr{C}}_{A,F} = R^{\mathscr{A}}_{A,B} \otimes R^{\mathscr{B}}_{B,F}.$$

Proof. Let $a_i a_j = \sum_k \gamma^k_{ij} a_k$ and $b_p b_q = \sum_q \varepsilon^r_{pq} b_r$, where $\gamma^k_{ij} \in B$ and $\varepsilon^r_{pq} \in F$. Clearly $\mathscr{C} = \mathscr{A}\mathscr{B}$ is a basis for A considered as an F-algebra, and

$$(a_i b_p)(a_j b_q) = a_i a_j b_p b_q = \left(\sum_k \gamma^k_{ij} a_k\right)\left(\sum_r \varepsilon^r_{pq} b_r\right) = \sum_{k,r} \gamma^k_{ij} \varepsilon^r_{pq} a_k b_r,$$

so the structure constants of A for the basis $\mathscr{C}$ over F are

$$\eta^{(k,r)}_{(i,p),(j,q)} = \gamma^k_{ij} \varepsilon^r_{pq}.$$

The stated results now follow as in the proof of Proposition 5. ∎

4. DETERMINATION OF THE RADICAL FROM L, R

For any fixed $c \in A$, define a map $\phi_c: A \to A$ by the rule $a\phi_c = ac$ for each $a \in A$. Then ϕ_c is linear, and so, with respect to any chosen basis for A as a vector space over F, we can represent ϕ_c by an $m \times m$ matrix with entries in F. Let $\tau(c)$ denote the trace of this matrix. Clearly this "right trace" map $\tau: A \to F$ is a linear functional, and the value of $\tau(c)$ does not depend on the choice of basis used.

LEMMA 1. *For any A, $\mathscr{A} = \{a_1,..., a_m\}$, we have*

$$\tau(a_k) = \sum_t \gamma^t_{tk} \qquad (k = 1,..., m),$$

$$\tau(a_i a_j) = \lambda^{\mathscr{A}}_{ij} \qquad (i, j = 1,..., m).$$

Proof. Using $\mathscr{A}$ as basis to obtain a matrix representation of the map ϕ_c, we find

$$\tau(c) = \text{trace}\ \phi_c = \sum_t (\text{coefficient of } a_t \text{ in } a_t c).$$

In particular,

$$\tau(a_k) = \sum_t (\text{coefficient of } a_t \text{ in } a_t a_k) = \sum_t \gamma_{tk}^t,$$

and so, by linearity,

$$\tau(a_i a_j) = \tau\left(\sum_k \gamma_{ij}^k a_k\right) = \sum_k \gamma_{ij}^k \tau(a_k) = \sum_k \gamma_{ij}^k \sum_t \gamma_{tk}^t = \lambda_{ij}^{\mathscr{A}}. \quad \blacksquare$$

Thus (in close analogy with the Killing form $\operatorname{tr}((\operatorname{ad} x)(\operatorname{ad} y))$ of a Lie algebra, or the trace form of a field extension) $L^{\mathscr{A}}$ is just the matrix representing the bilinear form $\tau(xy)$ with respect to the basis $\mathscr{A}$. Lemma 1 also provides an alternative proof of Proposition 1.

For suitably restricted F, we have the following criterion to determine whether a given element b of A lies in the nil radical N of A:

LEMMA 2. *Let $\mathscr{A} = \{a_1, ..., a_m\}$ be any basis for A over F, and let $b \in A$. Suppose also that F has characteristic either zero or a prime $p > m$.*

Then $b \in N$ iff $\tau(a_i b) = 0$ $(i = 1, ..., m)$.

This lemma is essentially well known (see [2, Chap. VII, especially pp. 106–107] or [3, p. 271, Lemma 2]). Again, there is a somewhat similar characterization of the radical of a Lie algebra in terms of the Killing form [5, p. 73, Theorem 5].

One can combine Lemmas 1 and 2 to obtain the following more explicit criterion (cf. [2, p. 108, Eq. (26)]):

THEOREM 1. *Let F be any field, A any finite-dimensional associative algebra over F, and $\mathscr{A} = \{a_1, ..., a_m\}$ any basis for A over F. Also assume that F has characteristic either zero or a prime $p > m$, let N denote the (nil) radical of A, and, for any given element $b \in A$, say $b = \beta_1 a_1 + \cdots + \beta_m a_m$ (where each $\beta_i \in F$), let $\mathbf{b}$ denote the column m-vector over F having components $\beta_1, ..., \beta_m$.*

Then $b \in N$ if and only if $L^{\mathscr{A}}\mathbf{b} = \mathbf{0}$; in particular, A is semisimple if and only if $L^{\mathscr{A}}$ is nonsingular.

Proof. For each i, we have, by linearity,

$$\tau(a_i b) = \tau\left(a_i \sum_j \beta_j a_j\right) = \sum_j \beta_j \tau(a_i a_j) = \sum_j \beta_j \lambda_{ij}^{\mathscr{A}} \tag{1}$$

by Lemma 1. But then, by Lemma 2, it follows that $b \in N$ iff $\sum_j \lambda_{ij}^{\mathscr{A}} \beta_j = 0$ $(i = 1, ..., m)$, i.e. iff $L^{\mathscr{A}}\mathbf{b} = \mathbf{0}$. $\blacksquare$

Of course, dually, we can characterize the radical equally by means of the right structure matrix, i.e., $b \in N$ iff $R^{\mathscr{A}}\mathbf{b} = \mathbf{0}$; and it follows at once from this that $L^{\mathscr{A}}, R^{\mathscr{A}}$ have the same row (and column) space. Also, as was pointed out in [3, Sect. 5, p. 275], in fact $b \in N$ implies $L^{\mathscr{A}}\mathbf{b} = R^{\mathscr{A}}\mathbf{b} = \mathbf{0}$ even without any restriction on the characteristic of F.

I have been unable to trace Lemma 2 or, equivalently Theorem 1, back to any source prior to Dickson [2], but, at least in the commutative case, the essential ideas were known to (and apparently discovered by) Dedekind.

5. Connections between L, R and the Structure of A/N

Given any F-algebra A, we can apply Theorem 1 to determine the radical $N = N(A)$ of A, and, by constructing a basis $\mathscr{B}$ of N (which, by Theorem 1, just amounts to reducing L to echelon form by row transformations), and extending this to a new basis $\mathscr{A}$ of A, we obtain, in the notation of Proposition 7, a basis $\mathscr{C} = \{a_1\theta, ..., a_m\theta\}$ of A/N, and also obtain the structure matrices $L^{\mathscr{C}}, R^{\mathscr{C}}$ of A/N relative to this basis.

It would obviously be desirable to be able to use $L^{\mathscr{C}}, R^{\mathscr{C}}$ to obtain the Wedderburn decomposition of the semisimple algebra A/N as the direct sum of its simple ideals $M_{n_s}(D_s)$, but unfortunately this is not in general possible; thus our "structure matrices" do not in fact determine the whole structure of A. The following result indicates the extent of this indeterminacy:

Proposition 9. *Let F be any quadratically closed field, let A be any semisimple F-algebra of finite dimension m, and assume that F has characteristic either zero or an odd prime $p > m$.*

Then there exists a basis $\mathscr{A}$ of A such that $L^{\mathscr{A}} = I_m$ (i.e., the $m \times m$ unity matrix).

Proof. For *any* basis of A, by Proposition 1 and Theorem 1 the corresponding L is symmetric and nonsingular, and so, by Proposition 2 (or its corollary), we can choose a basis $\mathscr{C} = \{c_1, ..., c_m\}$ for A such that $L^{\mathscr{C}}$ is diagonal with all its diagonal entries nonzero, say

$$L^{\mathscr{C}} = \mathrm{diag}(\mu_1, ..., \mu_m).$$

Also, by the quadratic closure of F, there exist nonzero $\xi_h \in F$ such that $\xi_h^2 = \mu_h$ $(h = 1, ..., m)$. But then, on writing $\mathscr{A} = \{\xi_1^{-1}c_1, ..., \xi_m^{-1}c_m\}$, we have $L^{\mathscr{A}} = I_m$ by Proposition 2. ∎

Indeed, by Theorem 4, in fact $L^{\mathscr{A}} = R^{\mathscr{A}} = I_m$.

Corollary. *For F as in Proposition 9, and any nonsingular symmetric $m \times m$ matrix L over F, there exists a semisimple F-algebra A and an F-basis $\mathscr{A}$ of A such that $L^{\mathscr{A}} = R^{\mathscr{A}} = L$.*

It also follows that, relative to suitable bases, the simple noncommutative algebra $A = M_2(F)$ has the same structure matrices L, R as does the nonsimple commutative algebra F^4; thus there can be *no* criterion, based solely on a knowledge of L, R, for simplicity or for commutativity (or, indeed, for any other property not determined by dimension alone in the semisimple case).

Of course the hypothesis of quadratic closure in Proposition 9 is essential (consider the group algebra $\mathbb{Q}[C_3]$ or $\mathbb{R}[C_3]$ of the cyclic group of order 3 over the rationals or reals).

Proposition 9 has the following extension to nonsemisimple algebras:

PROPOSITION 10. *Let F be any quadratically closed field, let A be any finite-dimensional F-algebra, say with $\dim_F A = m$, $\dim_F N(A) = n$, and assume that F has characteristic either zero or an odd prime $p > m$.*

Then there exists a basis $\mathscr{A}$ of A such that

$$L^{\mathscr{A}} = \begin{pmatrix} 0_n & 0 \\ 0 & I_{m-n} \end{pmatrix}.$$

Proof. Choose a basis $\{a_1, ..., a_n\}$ for $N(A)$, and extend this to a basis $\mathscr{B} = \{a_1, ..., a_n, a_{n+1}, ..., a_m\}$ for A. By Theorem 1, the first n columns of $L^{\mathscr{B}}$ must be zero, and so, by Proposition 1 and Theorem 1, we have $L^{\mathscr{B}} = \left(\begin{smallmatrix} 0 & 0 \\ 0 & M \end{smallmatrix}\right)$ for some nonsingular symmetric $(m-n) \times (m-n)$ matrix M. As above, it follows by Proposition 2 that there exists a new basis $\mathscr{C}$ of A such that

$$L^{\mathscr{C}} = \operatorname{diag}(0, ..., 0, \mu_{n+1}, ..., \mu_m),$$

with n zeros, and with $\mu_h \neq 0$ $(h = n+1, ..., m)$. Now argue as for Proposition 9. ∎

Clearly Propositions 9 and 10 limit the kinds of further applications of L, R (i.e., other than characterizing the radical) which can reasonably be hoped for. However, these limitations are not a total barrier to progress, and can sometimes be outflanked, e.g., in connection with structures present in *every* algebra, so that, in spite of Propositions 9 and 10, it may nevertheless be possible to use L, R to specify such structures explicitly:

THEOREM 2. *Let F, A, $\mathscr{A}$, $\mathbf{b}$ be as in Theorem 1, and define column m-vectors $\boldsymbol{\lambda} = \boldsymbol{\lambda}^{\mathscr{A}}$, $\boldsymbol{\rho} = \boldsymbol{\rho}^{\mathscr{A}}$ by*

$$\lambda_i = \sum_t \gamma_{ti}^t, \quad \rho_i = \sum_t \gamma_{it}^t \qquad (i = 1, ..., m).$$

Then, for any $b \in A$, we have $L\mathbf{b} = \boldsymbol{\lambda}$ iff $R\mathbf{b} = \boldsymbol{\rho}$ iff $b + N$ is the unity element of A/N.

Proof. Let $b=\beta_1 a_1+\cdots+\beta_m a_m \in A$ be such that $b+N$ is the unity of A/N. Then $ab+N=(a+N)(b+N)=a+N$, i.e., $ab \in a+N$ for every $a \in A$. Since τ vanishes on N, it follows that $\tau(ab)=\tau(a)$ for every $a \in A$, and so, by Eq. (1) in the proof of Theorem 1,

$$(L\mathbf{b})_i=\sum_j \lambda_{ij}\beta_j=\tau(a_i b)=\tau(a_i)=\sum_t \gamma_{ti}^t \qquad (i=1,\dots,m)$$

by Lemma 1, i.e., $L\mathbf{b}=\boldsymbol{\lambda}$.

Conversely, if $L\mathbf{b}=\boldsymbol{\lambda}$, consider the semisimple algebra A/N, having unity element $u+N$, say (of course $u+N$ is unique, but, if $N \neq 0$, then u is non-unique). Then, as above, $L\mathbf{u}=\boldsymbol{\lambda}$, so $L(\mathbf{b}-\mathbf{u})=\mathbf{0}$, i.e., $b-u \in N$ by Theorem 1, so that $b+N=u+N$ is the unity of A/N.

Thus $L\mathbf{b}=\boldsymbol{\lambda}$ iff $b+N$ is the unity of A/N; dually, also $R\mathbf{b}=\boldsymbol{\rho}$ iff $b+N$ is the unity of A/N. ∎

When L is nonsingular, then, writing $(L^{-1})_{ij}=\mu_{ij}$, one can verify that $b=\sum_{i,j}\mu_{ij}a_i a_j$ satisfies $L\mathbf{b}=\boldsymbol{\lambda}$, so that $\sum_{i,j}\mu_{ij}a_i a_j=1_A$. Thus, in Lie theory, an approach along the lines of Theorem 2 might yield a generalization, meaningful for nonsemisimple Lie algebras, of the Casimir operator determined by the Killing form.

For fields F which are not quadratically closed, of course Propositions 9 and 10 pose no threat. For example, we can explore the Wedderburn structure of A/N via the definiteness properties of L, R. We shall say that an $m \times m$ matrix L over F is *F-semidefinite* iff $\mathbf{b}'L\mathbf{b}=0$ implies $L\mathbf{b}=\mathbf{0}$ (where $\mathbf{b}$ denotes an arbitrary column m-vector over F, and $\mathbf{b}'$ its transpose). If L is diagonalizable by congruence transformations (as is always the case for symmetric L if F has characteristic zero or an odd prime), and if F is quadratically closed, then clearly L cannot be F-semidefinite unless L has rank 0 or 1. However, if F is a subfield of the reals, then our F-semidefiniteness concept reduces to the usual property of non-negative or nonpositive definiteness.

THEOREM 3. *Let F, A, N be as in Theorem* 1, *and assume that L is F-semidefinite. Then the quotient algebra A/N is isomorphic to a direct sum of division algebras over F.*

Proof. The argument is the same as for Corollary 3 in [3, p. 274]; as it is brief, we include it for completeness. For any basis $\mathscr{A}=\{a_1,\dots,a_m\}$ of A and any $b=\beta_1 a_1+\cdots+\beta_m a_m \in A$, we have

$$\tau(b^2)=\tau\left(\sum_{i,j}\beta_i\beta_j a_i a_j\right)=\sum_{i,j}\beta_i\beta_j\tau(a_i a_j)=\sum_{i,j}\lambda_{ij}\beta_i\beta_j=\mathbf{b}'L\mathbf{b}.$$

Thus, if $(b+N)^2$ is zero in A/N, then $b^2 \in N$, and so $\mathbf{b}'L\mathbf{b}=\tau(b^2)=0$, whence, by the semidefiniteness of L, we must have $L\mathbf{b} = \mathbf{0}$, i.e. (by Theorem 1), $b + N$ is the zero of A/N.

Hence A/N has no nonzero nilpotent element, so the stated conclusion follows from standard Wedderburn theory. ∎

Possibly some hypothesis weaker than F-semidefiniteness would suffice; in any case, Proposition 4 (or $\mathbb{Q}[C_3]$) shows that the converse of Theorem 3 is false.

6. Conclusion

Our discussions above leave some obvious questions unanswered: most notably, how far our various restrictions on F are essential (cf. [3, §5]), how to characterize those pairs (L, R) of symmetric $m \times m$ matrices over F which can arise as the structure matrices of an F-algebra A relative to a suitable basis $\mathscr{A}$, and (in connection with Proposition 2) whether (or for which F) we can always diagonalize $L^{\mathscr{A}}, R^{\mathscr{A}}$ simultaneously by choosing $\mathscr{A}$ suitably. Also L, R may turn out to have applications in directions not foreshadowed here. Meanwhile, we note an interesting fact which explains why $L = R$ in Propositions 3 and 4, and also focusses attention on a new class of algebras which may deserve further study.

Given an F-algebra A, we shall call A *dual* iff $L^{\mathscr{A}} = R^{\mathscr{A}}$ for some F-basis $\mathscr{A}$ of A; by Proposition 2, this implies $L^{\mathscr{C}} = R^{\mathscr{C}}$ for *every* F-basis $\mathscr{C}$ of A.

By Propositions 5 and 6, if two F-algebras A, B are both dual, then so are $A \otimes_F B$ and $A \oplus B$.

Theorem 4. *Every Frobenius algebra is dual.*

Proof. Let F be any field, let A be any Frobenius F-algebra, and write $m = \dim_F A$. Then (see e.g., [1, p. 424]) there exist two F-bases $\mathscr{A} = \{a_1, \ldots, a_m\}$, $\mathscr{B} = \{b_1, \ldots, b_m\}$ of A such that, for each $c \in A$, there exist corresponding scalars $\xi_{ij} = \xi_{ij}(c) \in F$ $(i, j = 1, \ldots, m)$ such that

$$a_i c = \sum_j \xi_{ij} a_j \qquad (i = 1, \ldots, m) \tag{2}$$

and

$$c b_i = \sum_j \xi_{ji} b_j \qquad (i = 1, \ldots, m). \tag{3}$$

Using (2) to evaluate $\tau(c)$ with respect to the basis $\mathscr{A}$, we find

$$\tau(c) = \text{trace } \phi_c = \sum_i (\text{coefficient of } a_i \text{ in } a_i c) = \sum_i \xi_{ii},$$

where it is important to recall that the same value for $\tau(c)$ would be obtained with respect to any other basis of A. Similarly, if we let $\sigma: A \to F$ denote the trace of the left regular representation of A, then, using (3) to evaluate $\sigma(c)$ with respect to $\mathscr{B}$, we find

$$\sigma(c) = \sum_i (\text{coefficient of } b_i \text{ in } cb_i) = \sum_i \xi_{ii} = \tau(c),$$

i.e., $\sigma = \tau: A \to F$.

Thus, for *any* F-basis $\mathscr{C} = \{c_1, ..., c_m\}$ of A, it follows from Proposition 1 and Lemma 1 (and its dual) that

$$\lambda_{ij}^{\mathscr{C}} = \tau(c_i c_j) = \sigma(c_i c_j) = \rho_{ji}^{\mathscr{C}} = \rho_{ij}^{\mathscr{C}},$$

i.e., $L^{\mathscr{C}} = R^{\mathscr{C}}$. ∎

It would be interesting to know whether the same is true for quasi-Frobenius algebras (e.g., Nakayama's 9-dimensional quasi-Frobenius, non-Frobenius algebra [6, p. 624] is dual).

COROLLARY. *Every semisimple algebra is dual.*

In other words, for *any* F, if either $L^{\mathscr{A}}$ or $R^{\mathscr{A}}$ is nonsingular, then they must be equal (since the parts of Lemma 2 and Theorem 1 involved do not depend on any assumptions about the characteristic of F, cf. [3, §5]). This generalizes Proposition 2 of [3].

From the last part of the proof of Theorem 4, clearly A is dual iff $\sigma = \tau$: $A^2 \to F$; in particular, if $A^2 = A$ (e.g., if A is an algebra with unity), then duality for A just means that the left and right regular representations have the same trace functional (i.e., on the whole of A).

The commutative case shows that the class of dual algebras includes the class of (quasi-) Frobenius algebras strictly. Theorem 4 also raises the question of exploring how far the voluminous theory of Frobenius algebras can be extended to dual algebras.

Of course, taking due care (by use of Proposition 2) to arrange for closure under isomorphism, one may use L, R to define, besides duality, many other new classes of algebras. However, it seems more promising to try to use L, R to characterize classes already of interest for other reasons (although, by Proposition 9, this approach seems unlikely, at least for certain F, to lead to anything of interest for semisimple algebras).

There seems to be no possibility of extending our structure matrix approach to infinite-dimensional algebras, since, for these (although, for fixed i, j, we should still have $\gamma_{ij}^k = 0$ for all sufficiently large k), for fixed k the formal sum $\sum_t \gamma_{tk}^t$ may contain infinitely many nonzero terms, so that λ_{ij} is not well-defined.

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